

NEW IMPROVED MOSER-TRUDINGER INEQUALITIES AND SINGULAR LIOUVILLE EQUATIONS ON COMPACT SURFACES

ANDREA MALCHIODI AND DAVID RUIZ

ABSTRACT. We consider a singular Liouville equation on a compact surface, arising from the study of Chern-Simons vortices in a self dual regime. Using new improved versions of the Moser-Trudinger inequalities (whose main feature is to be scaling invariant) and a variational scheme, we prove new existence results.

1. INTRODUCTION

We consider a compact orientable surface Σ with metric g and the equation

$$(1) \quad -\Delta_g u = \rho h(x)e^{2u} - 2\pi \sum_{j=1}^m \alpha_j \delta_{p_j} + c, \quad \int_{\Sigma} h(x)e^{2u} dV_g = 1.$$

Here ρ is a positive parameter, $h : \Sigma \rightarrow \mathbb{R}$ a smooth positive function, $\alpha_j \in [0, 1]$, $p_j \in \Sigma$ and c is a constant. Integrating by parts we get that a necessary condition for the existence of solution is $c = \left(2\pi \sum_{j=1}^m \alpha_j - \rho\right) |\Sigma|^{-1}$.

This equation arises from physical models such as the abelian Chern-Simons-Higgs theory and the Electroweak theory, see [32], [35], [36], [37]. We also refer to [56], [57], [59] and the bibliographies therein for a more recent and complete description of the subject. Here we limit ourselves to mention that u is related to the absolute value of the wave function in the above models, while the p_j 's, called *vortices*, are points where the wave function vanishes. Equation (1) has been the subject of several investigations, see for example [2], [3], [4], [6], [7], [9], [10], [16], [23], [30], [42], [51], [54], [55], [60].

Most of these results deal with asymptotic analysis or compactness of solutions, while relatively few ones are available concerning existence. In [24], [33] some perturbative results are given, providing solutions of multi-bump type for special values of the parameter ρ . In [10] an existence theorem is proved for surfaces with positive genus and for $\rho \in (4\pi, 8\pi)$. Finally, in [20] the Leray-Schauder degree is computed for $\alpha \geq 1$ and $\rho \in (4\pi, 8\pi)$, and so existence results are deduced.

Our goal is to develop a global variational theory for the equation, yielding existence of solutions under rather general conditions. In this paper we give a new improved Moser-Trudinger inequality, a basic tool for this strategy, and derive some first results in this spirit. Further existence results will be discussed in a forthcoming paper.

It is easy to see that equation (1) is equivalent to:

2000 *Mathematics Subject Classification.* 35B33, 35J35, 53A30, 53C21.

Key words and phrases. Geometric PDEs, Variational Methods, Min-max Schemes.

$$(2) \quad -\Delta_g u = \rho \left(\frac{h(x)e^{2u}}{\int_{\Sigma} h(x)e^{2u} dV_g} - \frac{1}{|\Sigma|} \right) - 2\pi \sum_{j=1}^m \alpha_j \left(\delta_{p_j} - \frac{1}{|\Sigma|} \right).$$

This last formulation has the advantage that it is invariant with respect to the addition of constants.

Let $G_p(x)$ be the Green's function of $-\Delta_g$ on Σ with singularity at p , namely the unique solution of

$$-\Delta_g G_p(x) = \delta_p - \frac{1}{|\Sigma|} \quad \text{on } \Sigma, \quad \text{with } \int_{\Sigma} G_p(x) dV_g = 0.$$

The change of variables

$$(3) \quad u \mapsto u + 2\pi \sum_{j=1}^m \alpha_j G_{p_j}(x)$$

transforms (2) into an equation of the form

$$(4) \quad -\Delta_g u = \rho \left(\frac{\tilde{h}(x)e^{2u}}{\int_{\Sigma} \tilde{h}(x)e^{2u} dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{on } \Sigma.$$

Since G_p has the asymptotic behavior $G_p(x) \simeq \frac{1}{2\pi} \log \frac{1}{d(x,p)}$ near p , we have

$$(5) \quad \tilde{h} > 0 \text{ on } \Sigma \setminus \cup_j \{p_j\}; \quad \tilde{h}(x) \simeq d(x, p_j)^{2\alpha_j} \quad \text{near } p_j.$$

Problem (4) is the Euler-Lagrange equation for the functional

$$(6) \quad I_{\rho}(u) = \int_{\Sigma} |\nabla_g u|^2 dV_g + 2 \frac{\rho}{|\Sigma|} \int_{\Sigma} u dV_g - \rho \log \int_{\Sigma} \tilde{h}(x) e^{2u} dV_g; \quad u \in H^1(\Sigma).$$

Recall the Moser-Trudinger inequality

$$(7) \quad \log \int_{\Sigma} e^{2(u-\bar{u})} dV_g \leq \frac{1}{4\pi} \int_{\Sigma} |\nabla_g u|^2 dV_g + C; \quad u \in H^1(\Sigma),$$

see e.g. [48]. From that inequality one can easily check that I_{ρ} is bounded from below if $\rho < 4\pi$. Moreover, 4π is a threshold value, in the sense that for larger values of ρ the functional does not have a finite lower bound. However one can still hope to find critical points of saddle type, using for example min-max schemes.

This strategy or other topological methods (jointly with blow-up estimates), have been used successfully for *regular Liouville equations* of the form (2) but with all the α_i 's equal to zero. Such problems have motivations arising from physics (study of mean field vorticity, or Chern-Simons theory without sources), or from conformal geometry (prescribing the Gauss curvature or some of its higher order counterparts), see [11], [12], [13], [15], [17], [18], [25], [26], [27], [28], [29], [31], [38], [39], [40], [43], [50], [53]. In the latter case, the function e^{2u} represents the conformal dilation of the background metric on a given surface. In fact, (2) also arises in the Gauss curvature prescription problem on surfaces with conical singularities: for more details we refer to [5].

In this framework, one main common tool for applying variational arguments is some kind of improvement of the Moser-Trudinger inequality. A classical example

is a result by J.Moser, [49], where he showed that the first constant in (7) can be taken to be $\frac{1}{8\pi}$ for even functions on S^2 . A more general improvement was obtained by T.Aubin in [1], still in the case of the standard sphere: he showed that for *balanced* metrics one can take any constant which is larger than $\frac{1}{8\pi}$ (provided C is taken large enough). Interesting applications were found for example in [15] where rather general conditions were given for prescribing the Gauss curvature on the sphere. Aubin's improvement was generalized by W.Chen and C.Li in [21] for all surfaces, under the condition that the conformal volume e^{2u} *spreads* into two distinct regions (separated by a positive distance). This result was used in [27] to produce solutions of the regular Liouville equation, see also [28], [29] and [45] for further progress on this direction.

The main goal of this paper is to obtain a new type of improved inequality and apply it to the study of (2). To explain the spirit of this improvement we recall the result in [58], which states that if \tilde{h} is as in (5) then the best constant \mathcal{A} for the inequality

$$\log \int_{\Sigma} \tilde{h} e^{2(u-\bar{u})} dV_g \leq \mathcal{A} \int_{\Sigma} |\nabla u|^2 dV_g + C \quad \left(\bar{u} = \int_{\Sigma} u dV_g \right)$$

is given by $(4\pi \min \{1, \min_i \{(1 + \alpha_i)^{-1}\}\})^{-1}$. Therefore if some of the α_i 's is negative the best possible constant is lower than $\frac{1}{4\pi}$, but if all the α_i 's are positive (as in our case) the best constant is just $\frac{1}{4\pi}$. One can easily see this by testing the inequality on a *standard bubble*, namely a function of the form

$$(8) \quad \varphi_{\lambda,x}(y) = \log \frac{\lambda}{1 + \lambda^2 \text{dist}(x,y)^2},$$

with center point x different from all the p_i 's. This function realizes the best constant in the regular case, and for the above choice of x there is basically no effect from the vanishing of \tilde{h} somewhere on Σ .

On the other hand, in [30] it was shown that for any $\alpha > -1$ there exists C_{α} such that

$$\log \int_B |x|^{2\alpha} e^{2(u-\bar{u})} dV_g \leq \frac{1}{4(1+\alpha)\pi} \int_B |\nabla_g u|^2 dV_g + C_{\alpha}; \quad u \in H_r^1(B).$$

In the latter formula B stands for the unit ball of \mathbb{R}^2 and H_r^1 denotes the space of radial functions of class H^1 in B . Our improvement substitutes the symmetry requirement with a condition which, heuristically, applies to a subset of functions having codimension two in H^1 , assuming $\alpha \in (0, 1]$. Roughly speaking, we associate to each function u a *center of mass* constructed out of the unit measure $\mu_u := \frac{|x|^{2\alpha} e^{2u}}{\int_B |x|^{2\alpha} e^{2u} dx}$: then the improvement would occur for functions whose center of mass is the origin (in our case, a singular point).

However, a rigorous proof of the above claim requires new arguments: the proof of Aubin (and in fact also Chen and Li's one) relies on the fact of being able to find two sets with positive distance (bounded away from zero) which both contain a finite portion of the total conformal volume. The positivity of the distance allows to find cutoff functions χ_i with bounded gradients and to apply the standard Moser-Trudinger inequality to $\chi_i u$, choosing then the $\chi_i u$ with the smaller Dirichlet energy. This strategy fails in our case since the measure μ_u can be arbitrarily concentrated near a single point.

What is needed in this context is a condition which stays invariant under dilation of the measure μ_u . To achieve it we use concentration functions (in the spirit of concentration-compactness results) and covering arguments through thick annuli, see Section 3 for details. Somehow, we want to define a *continuous* map from $H^1(\Sigma)$ into Σ which keeps track of the points with maximal concentration of conformal volume: an improvement will hold when the image of this map is one of the singularities. The new feature of our improvement is that it is scaling invariant.

As an application of the previous statement we have that if $\rho < 4\pi(1 + \alpha)$, then the above *center of mass* cannot coincide with the singularity if the energy $I_\rho(u)$ is sufficiently negative. The consequence of the above fact (with a proper localization near each singularity) is that low sublevels of the functional I_ρ inherit some topology from the surface Σ with a certain number of points removed. Using these considerations, we are able to prove the following result, where $G(\Sigma)$ denotes the genus of Σ .

Theorem 1.1. *Suppose $\alpha_j \in (0, 1]$ for all $j = 1, \dots, m$ and that $\rho \in (4\pi, 8\pi)$, with $\rho \neq 4\pi(1 + \alpha_j)$ for all j . Denote by J_ρ the subset of $\{p_1, \dots, p_m\}$ for which $\rho < 4\pi(1 + \alpha_i)$. Then if $(G(\Sigma), |J_\rho|) \neq (0, 1)$ problem (2) has a solution.*

Remark 1.2. *For $G(\Sigma) > 0$ this result has been proved in [10], Corollary 6. The case of the sphere is more delicate: in fact in [7] it is shown (via a Pohozaev identity) that on the standard sphere (S^2, g_0) (2) has no solution for $m = 1$ and $\rho \in (4\pi, 4\pi(1 + \alpha))$, $\alpha > 0$, which is precisely the case $(G(\Sigma), |J_\rho|) = (0, 1)$. Therefore, our condition $(G(\Sigma), |J_\rho|) \neq (0, 1)$ is somehow sharp.*

To prove Theorem 1.1 we use a min-max scheme which is performed in detail in Section 5. First we show that there exist test functions $\varphi_{\alpha, \lambda, x}$ (where α is a suitable parameter) which satisfy $I_\rho(\varphi_{\alpha, \lambda, x}) \rightarrow -\infty$ and $\mu_{\varphi_{\alpha, \lambda, x}} \rightarrow \delta_x$ as $\lambda \rightarrow -\infty$ whenever $x \in \Sigma \setminus J_\rho$. Notice that, by our assumptions, $\Sigma \setminus J_\rho$ is a non contractible set. We then consider continuous maps \mathfrak{h} from the *topological cone* over (a retraction of) $\Sigma \setminus J_\rho$ into $H^1(\Sigma)$, which coincide with the $\varphi_{\alpha, \lambda, x}$'s on the boundary of the cone. The lower bound on the functional described above shows that the supremum of I_ρ on the image of \mathfrak{h} has a uniform control from below, provided λ is sufficiently large. This allows us to show the admissibility of the variational class consisting of the above maps \mathfrak{h} (in this step the non contractibility of $\Sigma \setminus J_\rho$ is used), and to find Palais-Smale sequences for I_ρ at some bounded level.

At this point one can use a monotonicity result developed initially by Struwe, which consists in varying the parameter ρ and to show that for a sequence $\rho_n \rightarrow \rho$ there exist bounded Palais-Smale sequences and hence solutions to (2). The argument can then be completed using the a priori estimates of [10], which imply compactness of solutions for ρ belonging to the ranges in Theorem 1.1.

The plan of the paper is the following. In Section 2 we collect some preliminary results on the Moser-Trudinger inequality (plus some more or less known improvements), together with some compactness results and a deformation lemma. In Section 3 we use a covering argument to define a convenient center of mass for the measure μ_u (see the above notation). In Section 4 we obtain our new improved inequality, and lower bounds for I_ρ . Section 5 is devoted to the proof of Theorem 1.1; some final comments are given at the end.

Acknowledgements

A.M. has been supported by the project FIRB-Ideas *Analysis and Beyond*, and is grateful to the Mathematical Analysis department of the University of Granada for the kind hospitality. D.R. has been supported by the Spanish Ministry of Science and Innovation under Grant MTM2008-00988 and by J. Andalucía (FQM 116). He thanks SISSA for the hospitality during a stay in 2009. Both authors are grateful to D. Bartolucci and to the referee for some helpful comments.

2. NOTATION AND PRELIMINARIES

In this section we fix our notation and recall some useful known facts. We state in particular some variants and improvements of the Moser-Trudinger inequality, together with their consequences.

We write $\text{dist}(x, y)$ to denote the distance between two points $x, y \in \Sigma$. Moreover, the symbol $B_p(r)$ stands for the open metric ball of radius r and center p , and $A_p(r, R)$ the corresponding open annulus. $H^1(\Sigma)$ is the Sobolev space of the functions on Σ which are in $L^2(\Sigma)$ together with their first derivatives. The symbol $\|\cdot\|$ will denote the norm of $H^1(\Sigma)$. If Σ has boundary, $H_0^1(\Sigma)$ will denote the completion of $C_c^\infty(\Sigma)$ with respect to the Dirichlet norm. If $u \in H^1(\Sigma)$, $\bar{u} = \frac{1}{|\Sigma|} \int_\Sigma u dV_g = \oint_\Sigma u dV_g$ stands for the average of u . For a real number a we denote by I_ρ^a the set $\{u \in H^1(\Sigma) : I_\rho(u) \leq a\}$.

Large positive constants are always denoted by C , and the value of C is allowed to vary from formula to formula and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C , as C_δ , etc.. Also constants with this kind of subscripts are allowed to vary.

2.1. Improved Moser-Trudinger inequalities. We start by recalling the well known Moser-Trudinger inequality in a version that, when applied to the sphere, has also received the name of Onofri inequality (see e.g. [15]).

Proposition 2.1. *Let Σ be a compact surface. Then*

a) *If Σ has a boundary,*

$$(9) \quad \log \int_\Sigma e^{2u} dV_g \leq \frac{1}{2\pi} \int_\Sigma |\nabla_g u|^2 dV_g + 2 \oint_\Sigma u + C \quad \text{for every } u \in H^1(\Sigma).$$

b) *If Σ has a boundary,*

$$(10) \quad \log \int_\Sigma e^{2u} dV_g \leq \frac{1}{4\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C \quad \text{for every } u \in H_0^1(\Sigma).$$

c) *If Σ does not have a boundary, then*

$$(11) \quad \log \int_\Sigma e^{2u} dV_g \leq \frac{1}{4\pi} \int_\Sigma |\nabla_g u|^2 dV_g + 2 \oint_\Sigma u + C \quad \text{for every } u \in H^1(\Sigma).$$

The constant $\frac{1}{4\pi}$ in (11) is sharp, as one can see by using standard bubbles, peaked at some point of Σ (see (8)). The constant in (9) is instead multiplied by two, since one can center a bubble on the boundary of Σ , dividing approximatively by two both the conformal volume and the Dirichlet energy. In (10) we impose $u \in H_0^1(\Sigma)$, so that this phenomenon is ruled out and the constant becomes again 4π .

We begin by giving a *localized version* of the Moser-Trudinger inequality, following the ideas of [29].

Proposition 2.2. *Assume that Σ is a compact surface (with or without boundary), and $\tilde{h} : \Sigma \rightarrow \mathbb{R}$ measurable, $0 \leq \tilde{h}(x) \leq C_0$ a.e. $x \in \Sigma$. Let $\Omega \subset \Sigma$, $\delta > 0$ such that $\text{dist}(\Omega, \partial\Sigma) > \delta$.*

Then, for any $\varepsilon > 0$ there exists a constant $C = C(C_0, \varepsilon, \delta)$ such that for all $u \in H^1(\Sigma)$,

$$\log \int_{\Omega} \tilde{h}(x) e^{2u} dV_g \leq \frac{1}{4\pi - \varepsilon} \int_{\Sigma} |\nabla_g u|^2 dV_g + 2 \int_{\Sigma} u dV_g + C.$$

PROOF. We can assume that $\int_{\Sigma} u = 0$. Let us decompose

$$u = u_1 + u_2,$$

where $u_1 \in L^\infty(\Sigma)$ and $u_2 \in H^1(\Sigma)$ will be fixed later. We have

$$(12) \quad \log \int_{\Omega} \tilde{h}(x) e^{2(u_1+u_2)} dV_g \leq 2\|u_1\|_{L^\infty(\Omega)} + \log \int_{\Omega} \tilde{h}(x) e^{2u_2} dV_g + C.$$

We next consider a smooth cutoff function χ with values into $[0, 1]$ satisfying

$$\begin{cases} \chi(x) = 1 & \text{for } x \in \Omega, \\ \chi(x) = 0 & \text{if } \text{dist}(x, \Omega) > \delta/2, \end{cases}$$

and then define

$$\tilde{u}(x) = \chi(x) u_2(x).$$

Clearly, $\tilde{u} \in H_0^1(\Sigma)$, so we can apply inequality (10) to \tilde{u} , finding

$$\log \int_{\Omega} \tilde{h}(x) e^{2u_2} dV_g \leq \log \int_{\Sigma} e^{2\tilde{u}} dV_g + C \leq \frac{1}{4\pi} \int_{\Sigma} |\nabla(\chi(x) u_2(x))|^2 dV_g + C.$$

Using the Leibnitz rule and the Hölder inequality we obtain:

$$\int_{\Sigma} |\nabla(\chi(x) u_2(x))|^2 dV_g \leq (1 + \varepsilon) \int_{\Sigma} |\nabla u_2|^2 dV_g + C_\varepsilon \int_{\Sigma} |u_2|^2 dV_g.$$

From (12) and the last formulas we find

$$(13) \quad \log \int_{\Omega} \tilde{h}(x) e^{2u} dV_g \leq \frac{1 + \varepsilon}{4\pi} \int_{\Sigma} |\nabla u_2|^2 dV_g + C_\varepsilon \int_{\Sigma} |u_2|^2 dV_g + 2\|u_1\|_{L^\infty(\Omega)} + C.$$

To control the latter terms we use truncations in Fourier modes. Define V_ε to be the direct sum of the eigenspaces of the Laplacian on Σ (with Neumann boundary conditions) with eigenvalues less or equal than $C_\varepsilon \varepsilon^{-1}$. Take now u_1 to be the orthogonal projection of u onto V_ε . In V_ε the L^∞ norm is equivalent to the L^2 norm: by using Poincaré's inequality we get

$$C_\varepsilon \int_{\Sigma} |u_2|^2 dV_g \leq \varepsilon \int_{\Sigma} |\nabla u_2|^2 dV_g,$$

$$\|u_1\|_{L^\infty(\Omega)} \leq C'_\varepsilon \|u_1\|_{L^2(\Sigma)} \leq CC'_\varepsilon \left(\int_{\Sigma} |\nabla u_1|^2 dV_g \right)^{\frac{1}{2}} \leq \varepsilon \int_{\Sigma} |\nabla u_1|^2 dV_g + C''_\varepsilon.$$

Hence, from (13) and the above inequalities we derive the conclusion by renaming ε properly. ■

The next result, for $\tilde{h} = 1$ has been proved for the first time in [21]. Assuming \tilde{h} only bounded does not require any changes in the arguments of the proof. Roughly

speaking, it states that if the function e^{2u} is *spread* into two regions of Σ , then the constant in the Moser-Trudinger inequality can be basically divided by 2. Let us point out that the arguments for Proposition 2.2 could also be used to prove the following result:

Proposition 2.3. *Let Σ be a compact surface, $\tilde{h} : \Sigma \rightarrow \mathbb{R}$ with $0 \leq \tilde{h}(x) \leq C_0$. Let Ω_1, Ω_2 be subsets of Σ with $\text{dist}(\Omega_1, \Omega_2) \geq \delta_0$ for some $\delta_0 > 0$, and fix $\gamma_0 \in (0, \frac{1}{2})$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(C_0, \varepsilon, \delta_0, \gamma_0)$ such that*

$$\log \int_{\Sigma} \tilde{h}(x) e^{2u} dV_g \leq C + \frac{1}{8\pi - \varepsilon} \int_{\Sigma} |\nabla_g u|^2 dV_g + 2 \int_{\Sigma} u.$$

for all functions $u \in H^1(\Sigma)$ satisfying

$$(14) \quad \frac{\int_{\Omega_i} \tilde{h}(x) e^{2u} dV_g}{\int_{\Sigma} \tilde{h}(x) e^{2u} dV_g} \geq \gamma_0, \quad i = 1, 2.$$

A useful corollary of this result is the following one: for the proof see [21] or [27].

Corollary 2.4. *Suppose $\rho < 8\pi$. Then, given any $\varepsilon, r > 0$ there exists $L = L(\varepsilon, r) > 0$ such that*

$$I_{\rho}(u) \leq -L \quad \Rightarrow \quad \frac{\int_{B_x(r)} \tilde{h} e^{2u} dV_g}{\int_{\Sigma} \tilde{h} e^{2u} dV_g} > 1 - \varepsilon \quad \text{for some } x \in \Sigma.$$

2.2. Compactness of solutions and deformation lemma. Concerning (4), we have the following result, proved via blow-up analysis.

Theorem 2.5. ([10]) *Let Σ be a compact surface, and let u_i solve (4) with \tilde{h} as in (5), $\rho = \rho_i$, $\rho_i \rightarrow \bar{\rho}$, with $\alpha_j > 0$ and $p_j \in \Sigma$. Suppose that $\int_{\Sigma} \tilde{h} e^{2u_i} dV_g \leq C_1$ for some fixed $C_1 > 0$. Then along a subsequence u_{i_k} one of the following alternative holds:*

- (i): u_{i_k} is uniformly bounded from above on Σ ;
- (ii): $\max_{\Sigma} (2u_{i_k} - \log \int_{\Sigma} \tilde{h} e^{2u_{i_k}} dV_g) \rightarrow +\infty$ and there exists a finite blow-up set $S = \{q_1, \dots, q_l\} \in \Sigma$ such that
 - (a) for any $s \in \{1, \dots, l\}$ there exist $x_n^s \rightarrow q_s$ such that $u_{i_k}(x_n^s) \rightarrow +\infty$ and $u_{i_k} \rightarrow -\infty$ uniformly on the compact sets of $\Sigma \setminus S$,
 - (b) $\rho_{i_k} \frac{\int_{\Sigma} \tilde{h} e^{2u_{i_k}} dV_g}{\int_{\Sigma} \tilde{h} e^{2u_{i_k}} dV_g} \rightharpoonup \sum_{s=1}^l \beta_s \delta_{q_s}$ in the sense of measures, with $\beta_s = 4\pi$ for $q_s \notin \{p_1, \dots, p_m\}$, or $\beta_s = 4\pi(1 + \alpha_j)$ if $q_s = p_j$ for some $j = \{1, \dots, m\}$. In particular one has that

$$\bar{\rho} = 4\pi n + 4\pi \sum_{j \in J} (1 + \alpha_j),$$

for some $n \in \mathbb{N} \cup 0$ and $J \subseteq \{1, \dots, m\}$ (possibly empty) satisfying $n + |J| > 0$, where $|J|$ is the cardinality of the set J .

From the above result we obtain immediately the following corollary.

Corollary 2.6. *Suppose $\rho \in (4\pi, 8\pi)$, and that*

$$\rho \neq 4\pi(1 + \alpha_j) \quad \text{for all } j = 1, \dots, m.$$

Then the set of solutions of (4) with zero mean value is uniformly bounded in $C^2(\Sigma)$.

Our argument to prove existence of solutions relies on variational theory: more precisely, we look for some change in the topology of the sublevels of I_ρ which, via some deformation lemma, leads to the existence of critical points. The deformation lemma is usually employed when the *Palais-Smale condition holds*: this means that every sequence $(u_l)_l$ for which $I_\rho(u_l)$ converges and for which $I'_\rho(u_l)$ tends to zero would admit a converging subsequence. This condition allows to deform a sublevel into another, in case there are no critical points in between, following the (negative) gradient flow of the functional. The main role of the P-S condition is that the flow lines stay compact as long as their energy is bounded.

Unfortunately it is still unknown whether the P-S condition holds for I_ρ , and one has to bypass the argument via some other kind of compactness result. We present next the following lemma, obtained by M. Lucia ([43]) through a variation of an argument in [52] (see also [27]).

Lemma 2.7. ([43]) *Given $a, b \in \mathbb{R}$, $a < b$, the following alternative holds: either $\exists(\rho_l, u_l) \subseteq \mathbb{R} \times X$ satisfying*

$$I'_{\rho_l}(u_l) = 0 \text{ for every } l; \quad a \leq I_\rho(u_l) \leq b; \quad \rho_l \rightarrow \rho,$$

or the set I_ρ^a is a deformation retract of I_ρ^b .

In fact, the result in [43] was proved for the case of positive functions \tilde{h} , but it does extend to our case as well with substantially the same proof.

3. A COVERING ARGUMENT

In this section we use a covering argument (via thick annuli) to detect both a *concentration size* for the function $\tilde{h}e^{2u}$ and its location, which are useful to obtain (in the next section) the desired improved inequality.

Proposition 3.1. *Assume $\tilde{h} : \Sigma \rightarrow \mathbb{R}$, $0 \leq \tilde{h}(x) \leq C$, $\rho \in (4\pi, 8\pi)$ and take a constant $C_1 > 2$. There exist $\tau > 0$, $L_0 > 0$ and a continuous map*

$$\beta : I_\rho^{-L_0} \rightarrow \Sigma,$$

satisfying the following property: for any $u \in I_\rho^{-L_0}$ there exists $\bar{\sigma} > 0$ and $\bar{y} \in \Sigma$ such that $d(\bar{y}, \beta(u)) < 2C_1\bar{\sigma}$ and:

$$\int_{B_{\bar{y}}(\bar{\sigma})} \tilde{h}e^{2u} dV_g = \int_{\Sigma \setminus B_{\bar{y}}(C_1\bar{\sigma})} \tilde{h}e^{2u} dV_g \geq \tau \int_{\Sigma} \tilde{h}e^{2u} dV_g.$$

PROOF. Let us define:

$$\mathcal{A} = \{f \in L^1(\Sigma), f(x) > 0 \text{ a.e.}, \int_{\Sigma} f dV_g = 1\},$$

$$\sigma : \Sigma \times \mathcal{A} \rightarrow (0, +\infty),$$

where $\sigma = \sigma(x, f)$ is chosen such that:

$$\int_{B_x(\sigma)} f dV_g = \int_{\Sigma \setminus B_x(C_1 \sigma)} f dV_g.$$

It is easy to check that $\sigma(x, f)$ is uniquely determined and continuous. Moreover, we obtain that σ satisfies

$$(15) \quad \text{dist}(x, y) \leq C_1 \max\{\sigma(x, f), \sigma(y, f)\} + \min\{\sigma(x, f), \sigma(y, f)\}.$$

Otherwise, $B_x(C_1 \sigma(x, f)) \cap B_y(\sigma(y, f) + \varepsilon) = \emptyset$ for some $\varepsilon > 0$. Let us now show that $A_y(\sigma(y, f), \sigma(y, f) + \varepsilon)$ is a nonempty open set. Clearly, $B_y(\sigma(y, f) + \varepsilon)$ does not exhaust the whole surface Σ . Since Σ is connected, there exists $z \in \partial B_y(\sigma(y, f) + \varepsilon)$. Then $\text{dist}(z, y) = \sigma(y, f) + \varepsilon$, which implies that $B_z(\varepsilon) \cap B_y(\sigma(y, f) + \varepsilon)$ is a nonempty open set included in $A_y(\sigma(y, f), \sigma(y, f) + \varepsilon)$.

Then:

$$\int_{B_x(\sigma(x, f))} f dV_g = \int_{\Sigma \setminus B_x(C_1 \sigma(x, f))} f dV_g \geq \int_{B_y(\sigma(y, f) + \varepsilon)} f dV_g > \int_{B_y(\sigma(y, f))} f dV_g.$$

By interchanging the roles of x and y , we would also obtain the reverse inequality. This contradiction proves (15).

Let us define $T : \Sigma \times \mathcal{A} \rightarrow (0, +\infty)$ by

$$T(x, f) = \int_{B_x(\sigma(x, f))} f dV_g.$$

Clearly, T is also continuous.

Step 1: There exists $\tau > 0$ such that $\max_{x \in \Sigma} T(x, f) > 2\tau$ for any $f \in \mathcal{A}$.

Let us take $x_0 \in \Sigma$ such that $T(x_0, f) = \max_{x \in \Sigma} T(x, f)$, and fix some $x \in A_{x_0}(\sigma(x_0, f), C_1 \sigma(x_0, f))$.

We claim that:

$$(16) \quad \text{dist}(x, x_0) + C_1 \sigma(x, f) \geq C_1 \sigma(x_0, f),$$

$$(17) \quad \text{dist}(x, x_0) - C_1 \sigma(x, f) \leq \sigma(x_0, f).$$

Let us prove (16). By contradiction, assume $\text{dist}(x, x_0) + C_1 \sigma(x, f) < C_1 \sigma(x_0, f) - 2\varepsilon$ for some $\varepsilon > 0$; by the triangular inequality, $B_x(C_1 \sigma(x, f)) \subset B_{x_0}(C_1 \sigma(x_0, f) - 2\varepsilon)$. By definition of σ , $B_{x_0}(C_1 \sigma(x_0, f)) \neq \Sigma$. So we can show, as previously, that $A_{x_0}(C_1 \sigma(x_0, f) - 2\varepsilon, C_1 \sigma(x_0, f))$ is not empty. Then:

$$\begin{aligned} T(x, f) &= \int_{B_x(\sigma(x, f))} f dV_g = \int_{\Sigma \setminus B_x(C_1 \sigma(x, f))} f dV_g \\ &> \int_{\Sigma \setminus B_{x_0}(C_1 \sigma(x_0, f))} f dV_g = T(x_0, f), \end{aligned}$$

which contradicts the definition of x_0 .

We now prove (17) in an analogous way. Indeed, if $\text{dist}(x, x_0) - C_1 \sigma(x, f) > \sigma(x_0, f) + 2\varepsilon$, we obtain that $(\Sigma \setminus B(x, C_1 \sigma(x, f))) \supset B(x_0, \sigma(x_0, f) + 2\varepsilon)$. As above, the open set $A_{x_0}(\sigma(x_0, f), \sigma(x_0, f) + \varepsilon)$ is nonempty. Then, we obtain again a contradiction:

$$T(x, f) = \int_{B_x(\sigma(x, f))} f dV_g = \int_{\Sigma \setminus B_x(C_1 \sigma(x, f))} f dV_g > \int_{B_{x_0}(\sigma(x_0, f))} f dV_g = T(x_0, f).$$

The claim is proved.

We subtract (17) from (16), and deduce that $\sigma(x, f) \geq \frac{C_1-1}{2C_1}\sigma(x_0, f) \geq \frac{1}{4}\sigma(x_0, f)$ for any $x \in A_{x_0}(\sigma(x_0, f), C\sigma(x_0, f))$.

We now point out that given $C_1 > 2$, there exists $k = k(C_1)$ such that, for any $\sigma > 0$ and any $y \in \Sigma$,

$$A_y(\sigma, C_1\sigma) \subset \cup_{i=1}^k B(x_i, \frac{1}{4}\sigma),$$

for some $x_i \in A_y(\sigma, C_1\sigma)$.

Therefore:

$$\int_{A_{x_0}(\sigma(x_0, f), C_1\sigma(x_0, f))} f dV_g \leq \sum_{i=1}^k \int_{B_{x_i}(\sigma(x_i, f))} f dV_g = \sum_{i=1}^k T(x_i, f) \leq k T(x_0, f).$$

On the other hand:

$$\int_{B_{x_0}(\sigma(x_0, f))} f dV_g = \int_{\Sigma \setminus B_{x_0}(C_1\sigma(x_0, f))} dV_g = T(x_0, f).$$

Hence $1 = \int_{\Sigma} f dV_g \leq (k+2)T(x_0, f)$, which concludes the proof of Step 1.

Step 2: Definition of $\bar{\sigma}$ and \bar{y} .

Let us define:

$$S(f) = \{x \in \Sigma : T(x, f) \geq \tau\}.$$

By Step 1, $S(f)$ is a nonempty compact set for every $f \in \mathcal{A}$. Let us define also:

$$\bar{\sigma}(f) = \max_{x \in S(f)} \sigma(x, f).$$

Observe that, in general, $\bar{\sigma}$ could be discontinuous in f . Finally, take $\bar{y} \in S(f)$ such that $\sigma(\bar{y}, f) = \bar{\sigma}$. For $u \in I_{\rho}^{-L}$, take $f = \left(\int_{\Sigma} \tilde{h} e^{2u} dV_g \right)^{-1} \tilde{h}(x) e^{2u(x)} = \tilde{h}(x) e^{2u(x) - \log \int_{\Sigma} \tilde{h} e^{2u} dV_g} \in \mathcal{A}$.

Step 3: For any $\varepsilon > 0$ there exists $L_0 > 0$ large enough such that $\text{diam } S(f) \leq (C_1 + 1)\bar{\sigma} < \varepsilon$ for $u \in I_{\rho}^{-L_0}$.

The first inequality holds independently of L_0 ; indeed, by (15), $\text{dist}(x, y) \leq (C_1 + 1)\bar{\sigma}$ for any given any $x, y \in S(f)$.

On the other hand, recall that:

$$\begin{aligned} \int_{B_{\bar{y}}(\bar{\sigma})} \tilde{h} e^{2u} dV_g &\geq \tau \int_{\Sigma} \tilde{h} e^{2u} dV_g, \text{ and} \\ \int_{\Sigma \setminus B_{\bar{y}}(C_1\bar{\sigma})} \tilde{h} e^{2u} dV_g &\geq \tau \int_{\Sigma} \tilde{h} e^{2u} dV_g. \end{aligned}$$

Corollary 2.4 implies that we can choose $L_0 > 0$ so that $\bar{\sigma} < \frac{\varepsilon}{C_1+1}$ for any $u \in I_{\rho}^{-L_0}$.

Step 4: Definition of $\beta(u)$ and conclusion.

We can assume to have an embedding $\Sigma \subset \mathbb{R}^3$. Let us define:

$$\eta : I_\rho^{-L_0} \rightarrow \mathbb{R}^3, \quad \eta(u) = \frac{\int_\Sigma [T(x, f) - \tau]^+ x dV_g}{\int_\Sigma [T(x, f) - \tau]^+ dV_g} \quad \text{where } f = \tilde{h}(x) e^{2u(x) - \log \int_\Sigma \tilde{h} e^{2u} dV_g}.$$

Observe that in the above terms the integrands are equal to zero outside $S(f)$.

Let $U \supset \Sigma$, $U \subset \mathbb{R}^3$ an open tubular neighborhood of Σ , and $P : U \rightarrow \Sigma$ an orthogonal projection onto Σ . Thanks to Step 3, $S(f) \subset \bar{B}_{\bar{y}}((C_1 + 1)\bar{\sigma}) \subset \bar{B}_{\bar{y}}^{\mathbb{R}^3}((C_1 + 1)\bar{\sigma})$. Since $\eta(u)$ is a barycenter of a function supported in $S(f)$, we have:

$$(18) \quad |\eta(u) - \bar{y}| \leq (C_1 + 1)\bar{\sigma}.$$

By taking $L_0 > 0$ large enough, $\eta(u) \in U$ for any $u \in I_\rho^{-L_0}$. Therefore, we can define:

$$\beta : I_\rho^{-L_0} \rightarrow \Sigma, \quad \beta(u) = P \circ \eta(u).$$

To conclude the proof we just need to show that $\text{dist}(\beta(u), \bar{y}) < 2C_1\bar{\sigma}$. We denote by $T_{\bar{y}}\Sigma$ the tangent space to Σ at \bar{y} . For any $x \in S(f) \subset B_{\bar{y}}((C_1 + 1)\bar{\sigma})$, one has:

$$(19) \quad \min\{|\bar{y} + y - x| : y \in T_{\bar{y}}\Sigma\} \leq C\bar{\sigma}^2,$$

where C depends only on the C^2 regularity of Σ . Since $\eta(u)$ is a barycenter of a function supported in $S(f)$, again, we have that

$$\eta(u) \in \bar{B}_{\bar{y}}^{\mathbb{R}^3}((C_1 + 1)\bar{\sigma}), \quad \min\{|\bar{y} + y - \eta(u)| : y \in T_{\bar{y}}\Sigma\} \leq C\bar{\sigma}^2.$$

By taking a larger L_0 , if necessary, we can assume $2C\bar{\sigma}^2 \leq \bar{\sigma}$ (recall, again, Step 3). So,

$$|\beta(u) - \eta(u)| = \min_{x \in \Sigma} |\eta(u) - x| \leq 2C\bar{\sigma}^2 \leq \bar{\sigma}.$$

This inequality, together with (18), implies that

$$(20) \quad |\beta(u) - \bar{y}| \leq (C_1 + 2)\bar{\sigma}.$$

Finally, take $\nu = \frac{2C_1}{C_1 + 2} > 1$; by Step 3 we can take L_0 larger, if necessary, so that $\bar{\sigma}$ verifies that for any $x, y \in \Sigma$, if $|x - y| \leq (C_1 + 2)\bar{\sigma}$, then $\text{dist}(x, y) \leq \nu|x - y|$. This, together with (20), finishes the proof. ■

Remark 3.2. We claim that, with the above construction, if $f_n = \frac{\tilde{h}e^{2u_n}}{\int_\Sigma \tilde{h}e^{2u_n} dV_g} \rightharpoonup \delta_x$ for some $x \in \Sigma$ then one also has $\beta(u_n) \rightarrow x$.

To check this, take $\bar{\sigma}_n = \bar{\sigma}(f_n)$, and $x_n \in S(f_n)$. Passing to a subsequence, we have $\bar{\sigma}_n \rightarrow \sigma_0$ and $x_n \rightarrow x_0$.

We first prove that $\bar{\sigma}_0 = 0$. If not, by construction, we would have:

$$\int_{B(x_0, \sigma_0/2)} f_n dV_g > \tau, \quad \int_{\Sigma \setminus B(x_0, (C_1 - 1)\sigma_0)} f_n dV_g > \tau,$$

which is a contradiction.

We now prove that $x_0 = x$. If not, take $0 < \delta < \text{dist}(x_0, x)/2$. Since $\bar{\sigma}_n \rightarrow 0$,

$$\int_{B(x_0, \delta)} f_n dV_g > \tau,$$

which is again a contradiction.

Recall now that, by (20), $\text{dist}(\beta(u_n), S(f_n)) \leq C\bar{\sigma}_n$; this concludes the proof of the claim.

Remark 3.3. If $\Sigma = B_0(1) \in \mathbb{R}^2$ and u is a radial function, then also f is a radial function. Moreover, $\sigma(x, f)$ and $T(x, f)$ are radial functions, and the set $S(f)$ is radially symmetric. Therefore, $\eta(u) = 0$, and here the projection β coincides with η . So, any radial function has zero barycenter.

This argument applies also to less restrictive symmetry assumptions: for instance, if u is even with respect to both the x and the y axes.

Observe that in this case β is defined in $H^1(B_0(1))$, and not only on sublevels.

4. IMPROVED INEQUALITIES

The main goal of this section is to prove the following result, giving a lower bound on the functional I_ρ under suitable conditions on its argument. Let β be the map constructed in Proposition 3.1.

Proposition 4.1. Assume $\rho \in (4\pi, 4\pi(1 + \alpha_i))$, $C_1 > 1$ sufficiently large, $L_0 > 0$, $\tau > 0$ such that Proposition 3.1 applies. Then, there exists $L > L_0$ such that $I_\rho(u) > -L$ for any $u \in I_\rho^{-L_0}$ satisfying that $\beta(u) = p_i$.

Remark 4.2. In particular, if $\Sigma = B_0(1) \subset \mathbb{R}^2$, $p = 0$ and u is radially symmetric, we obtain the boundedness below of I_ρ for $\rho < 4\pi(1 + \alpha)$ (see Remark 3.3). This has already been observed in [30].

The same thing is true under less restrictive symmetry assumptions: for instance, for functions u that are even with respect to both the x and the y axes (see, again, Remark 3.3).

PROOF. Take $\delta > 0$ fixed, $u \in I_\rho^{-L_0}$ such that $\beta(u) = p_i$, and let $\bar{y} \in \Sigma$, $\bar{\sigma} > 0$ be as in Proposition 3.1. For simplicity, let us assume that $\int_\Sigma u dV_g = 0$.

Take $\varepsilon > 0$ (to be fixed later), and choose $s \in (2\bar{\sigma}, \frac{C_1}{2}\bar{\sigma})$ such that:

$$(21) \quad \int_{A_{\bar{y}}(s/2, 2s)} |\nabla u|^2 dV_g < \frac{1}{\gamma} \int_{B_{\bar{y}}(\delta)} |\nabla u|^2 dV_g, \quad \text{where } \gamma \in \mathbb{N}, \quad \varepsilon^{-1} < \gamma \leq \frac{\log_2 C_1}{2}.$$

Let us define:

$$\begin{aligned} \mathcal{D}_1 &= \int_{B_{\bar{y}}(s)} |\nabla u|^2 dV_g, \quad \mathcal{D}_2 = \int_{\Sigma \setminus B_{\bar{y}}(s)} |\nabla u|^2 dV_g, \\ \mathcal{D} &= \mathcal{D}_1 + \mathcal{D}_2, \quad \mathcal{J} = \log \int_\Sigma \tilde{h}(x) e^{2u} dV_g. \end{aligned}$$

Here the proof proceeds in three steps:

Step 1: We apply Proposition 2.2 to u or, more precisely, to a convenient dilation of u given by:

$$v(x) = u(sx + \bar{y}).$$

We have:

$$\begin{aligned} \int_{B(\bar{y}, s)} |\nabla u|^2 dV_g &= \int_{B(0, 1)} |\nabla v|^2 dV_g, \\ \int_{B(\bar{y}, s)} u dV_g &= \int_{B(0, 1)} v dV_g, \end{aligned}$$

$$\begin{aligned} \int_{B(\bar{y}, s/2)} \tilde{h}(x) e^{2u} dV_g &\leq C \int_{B(\bar{y}, s/2)} |x - p_i|^{2\alpha_i} e^{2u} dV_g \leq C s^{2\alpha_i} \int_{B(\bar{y}, s/2)} e^{2u} dV_g = \\ &C s^{2+2\alpha_i} \int_{B(0, 1/2)} e^{2v} dV_g. \end{aligned}$$

In the above computations we have used that $|\bar{y} - p_i| \leq Cs$. By applying Proposition 2.2 to v and taking into account the inequality $\int_{B_{\bar{y}}(s/2)} \tilde{h} e^{2u} \geq \tau \int_{\Sigma} \tilde{h} e^{2u}$ (recall Proposition 3.1), we obtain:

$$(22) \quad \mathcal{J} - \frac{1}{4\pi - \varepsilon} \mathcal{D}_1 + 2(1 + \alpha_i) \log(1/s) \leq 2 \int_{B(\bar{y}, s)} u + C.$$

This inequality is one of the key ingredients of the proof.

Step 2: We estimate $\int_{\partial B_{\bar{y}}(s)} u$. To begin, consider a fixed value s and the function $\tilde{u} = u - \int_{B_{\bar{y}}(s)} u$. By the trace embedding, $\tilde{u} \in L^1(\partial B_{\bar{y}}(s))$, and thanks to the Poincaré-Wirtinger inequality we get

$$\left| \int_{\partial B_{\bar{y}}(s)} \tilde{u} dx \right| \leq C \|\tilde{u}\|_{H^1} \leq C \left(\int_{B(\bar{y}, s)} |\nabla u|^2 dV_g \right)^{1/2}.$$

Therefore,

$$(23) \quad \left| \int_{\partial B_{\bar{y}}(s)} u dV_g - \int_{B_{\bar{y}}(s)} u dV_g \right| \leq C \left(\int_{B(\bar{y}, s)} |\nabla u|^2 dV_g \right)^{1/2} \leq \varepsilon \mathcal{D}_1 + C'.$$

Observe now that the above inequality is invariant under dilation. So, the constant C is independent of s , and hence C' depends only on ε .

Step 3: By taking into account that $\tilde{h}(x) \sim d(x, p_i)^{2\alpha_i}$ near p_i , and $|x - p_i| \leq C|x - \bar{y}|$, we get the following estimate:

$$(24) \quad \int_{\Sigma \setminus B_{\bar{y}}(s)} \tilde{h}(x) e^{2u} dV_g = \int_{\Sigma \setminus B_{\bar{y}}(s)} \frac{\tilde{h}(x)}{|x - \bar{y}|^{2\alpha_i}} |x - \bar{y}|^{2\alpha_i} e^{2u} dV_g \leq \frac{C}{s^{2\alpha_i}} \int_{\Sigma \setminus B_{\bar{y}}(s)} e^{2v} dV_g,$$

with $v(x) = \hat{u}(x) + 2\alpha_i w(x)$,

$$w(x) = \begin{cases} \log s & x \in B_{\bar{y}}(s), \\ \log |x| & x \in A_{\bar{y}}(s, \delta) \\ \log \delta & x \in \Sigma \setminus B_{\bar{y}}(\delta) \end{cases}, \quad \begin{cases} -\Delta_g \hat{u} = 0 & x \in B(\bar{y}, s), \\ \hat{u}(x) = u(x) & x \notin B(\bar{y}, s). \end{cases}$$

Our intention is to apply the Moser-Trudinger to v . Observe that:

$$\int_{\Sigma} v \leq C + \int_{\Sigma} \hat{u}.$$

Since $\int_{\Sigma} u = 0$ and $\hat{u} - u$ has compact support in $B_{\bar{y}}(s)$,

$$\left| \int_{\Sigma} \hat{u} \right| = \left| \int_{\Sigma} (\hat{u} - u) \right| \leq C \left(\int_{B_{\bar{y}}(s)} |\nabla \hat{u} - \nabla u|^2 dV_g \right)^{1/2} \leq \varepsilon \mathcal{D} + C_{\varepsilon}.$$

We now estimate the Dirichlet energy:

$$\int_{B_{\bar{y}}(s)} |\nabla v|^2 dV_g = \int_{B_{\bar{y}}(s)} |\nabla \hat{u}|^2 dV_g \leq C_0 \int_{A_{\bar{y}}(s/2, 2s)} |\nabla u|^2 dV_g < C_0 \varepsilon \mathcal{D},$$

where C_0 is a universal constant. Moreover,

$$\begin{aligned} \int_{\Sigma \setminus B_{\bar{y}}(s)} |\nabla v|^2 dV_g &= \int_{\Sigma \setminus B_{\bar{y}}(s)} |\nabla u|^2 dV_g + 4\alpha_i^2 \int_{\Sigma \setminus B_{\bar{y}}(s)} \frac{1}{|x - \bar{y}|^2} dV_g \\ &\quad + 4\alpha_i \int_{\Sigma \setminus B_{\bar{y}}(s)} \nabla u \cdot \nabla (\log |x - \bar{y}|) dV_g. \end{aligned}$$

We integrate by parts to obtain:

$$\int_{\Sigma \setminus B_{\bar{y}}(s)} |\nabla v|^2 dV_g \leq \mathcal{D}_2 + 8\pi\alpha_i^2 \log \frac{1}{s} - 8\pi\alpha_i \oint_{\partial B_{\bar{y}}(s)} u dV_g + C.$$

Next, we apply the Moser-Trudinger inequality to v :

$$\mathcal{J} \leq 2\alpha_i \log \frac{1}{s} + \frac{1}{4\pi} \mathcal{D}_2 + 2\alpha_i^2 \log \frac{1}{s} - 2\alpha_i \oint_{\partial B_{\bar{y}}(s)} u dV_g + C_0 \varepsilon \mathcal{D} + C'.$$

By using (22) and (23), we obtain:

$$(1 + \alpha_i) \mathcal{J} \leq \frac{\alpha_i}{4\pi - \varepsilon} \mathcal{D}_1 + \frac{1}{4\pi} \mathcal{D}_2 + C_0 \varepsilon \mathcal{D} + \varepsilon \mathcal{D}_1 + C''.$$

In order to finish the proof it suffices to take ε small enough depending only on ρ and C_0 , (we recall that C_0 is a universal constant). In such case, we need to choose C_1 large enough so that (21) can be satisfied. ■

Remark 4.3. *The inequality used in (24) is sharp when evaluated on solutions of the singular equation $-\Delta u = |x|^{2\alpha} e^{2u}$ in \mathbb{R}^2 (and properly glued on Σ), which have been classified in [51] (see also [16]) as*

$$\varphi_{\alpha, \lambda}(y) = \log \frac{\lambda^{\alpha+1}}{(1 + (\lambda|x|)^{2(1+\alpha)})}.$$

If u has this form, and if $v = u + 2\alpha \log |x|$, then v has the asymptotic profile of a standard bubble, which makes the coefficients in the standard Moser-Trudinger inequality optimal.

Let J_ρ be as in Theorem 1.1, and given a small positive number θ we define the compact set

$$(25) \quad \Theta_\rho = \Sigma \setminus \cup_{p_i \in J_\rho} B_{p_i}(\theta).$$

From an extension of the above arguments one can prove the following result.

Proposition 4.4. *For $L > 0$ sufficiently large there exists a continuous projection Ψ from I_ρ^{-L} into Θ_ρ with the property that if $\frac{\bar{h}e^{2u_n}}{\int_\Sigma \bar{h}e^{2u_n} dV_g} \rightharpoonup \delta_x$ for some $x \in \Theta_\rho$ then $\Psi(u_n) \rightarrow x$.*

PROOF. It is sufficient to modify properly the function β constructed in Proposition 3.1. Let us notice that since $\rho < 4\pi(1 + \alpha_i)$ for $p_i \in J_\rho$, if $\beta(u) = p_i$, $p_i \in J_\rho$, then by Proposition 4.1 (and the subsequent observation) I_ρ is uniformly bonded from below. It follows that if L is sufficiently large and if $u \in I_\rho^{-L}$, then $\beta(u) \in \Sigma \setminus J_\rho$.

Then, if $\beta(u) \notin \Theta_\rho$, it will belong to some set of the form $B_{p_i}(\theta) \setminus \{p_i\}$, for some $p_i \in J_\rho$. At this point it is sufficient to move $\beta(u)$ along the geodesic segment

emanating from p_i in the direction of $\beta(u)$ until we hit the boundary of Θ_ρ . This procedure is well defined if θ is chosen sufficiently small.

The last statement of the proposition follows from Remark 3.2. ■

5. PROOF OF THEOREM 1.1 AND FINAL REMARKS

Let $\rho, J_\rho \subseteq \{p_1, \dots, p_m\}$ be as in Theorem 1.1, and let Θ_ρ be defined as in (25).

By our assumptions, the set Θ_ρ has either the topology of S^2 , or the topology of S^2 with more than one point removed, or the topology of any other surface with k points removed, $k \geq 0$. In any of these situations, the set Θ_ρ is non contractible.

We show next that it is possible to map an image of Θ_ρ homeomorphically into arbitrarily low sublevels of I_ρ . Let $\tilde{\alpha} = \max_{i \in \{1, \dots, m\} \setminus J_\rho} \alpha_i$. For $\lambda > 0$ and $\alpha \in (\tilde{\alpha}, \frac{\rho}{4\pi} - 1)$, we consider the following function

$$(26) \quad \varphi_{\alpha, \lambda, x}(y) = \log \left(\frac{\lambda^{1+\alpha}}{1 + (\lambda \text{dist}(y, x))^{2(1+\alpha)}} \right).$$

Lemma 5.1. *Let $\varphi_{\alpha, \lambda, x}$ be as in (26). Then one has*

$$I_\rho(\varphi_{\alpha, \lambda, x}) \rightarrow -\infty \quad \text{as } \lambda \rightarrow +\infty$$

uniformly on Θ_ρ .

PROOF. It is standard (see e.g. Section 4 in [46]) to check that

$$(27) \quad \int_{\Sigma} |\nabla \varphi_{\alpha, \lambda, x}|^2 dV_g \simeq 8\pi(1+\alpha)^2 \log \lambda; \quad \int_{\Sigma} \varphi_{\alpha, \lambda, x} dV_g \simeq -(1+\alpha) \log \lambda$$

as $\lambda \rightarrow +\infty$. We want to estimate next the integral of the exponential term. If x belongs to a compact set of $\Sigma \setminus \{p_1, \dots, p_m\}$ then for some $\delta > 0$ we have that

$$(28) \quad \int_{\Sigma} \tilde{h} e^{2\varphi_{\alpha, \lambda, x}} dV_g \geq \frac{1}{C_\delta} \int_{B_x(\delta)} e^{2\varphi_{\alpha, \lambda, x}} dV_g.$$

Using normal geodesic coordinates y centered at x and the estimate

$$(29) \quad dV_g = (1 + o_\delta(1)) dy; \quad d(x, y) = (1 + o_\delta(1))|y|, \quad y \in B_x(\delta)$$

(where we identified y with its coordinate in the above system) we find that

$$\int_{B_x(\delta)} e^{2\varphi_{\alpha, \lambda, x}} dV_g = (1 + o_\delta(1)) \int_{B_0(\delta)} \frac{\lambda^{2(1+\alpha)}}{(1 + (\lambda|y|)^{2(1+\alpha)})^2} dy \geq \frac{1}{C} \lambda^{2\alpha}.$$

The last formula, with (27), (28) and the fact that $\rho > 4\pi(1+\alpha)$ imply the conclusion for x in compact sets of $\Sigma \setminus \{p_1, \dots, p_m\}$.

Next, it is sufficient to consider the case in which x is close to one of the p_i 's with $i \notin J_\rho$. Localizing the integral as in (28) and (29), it is sufficient to estimate from below the following quantity

$$\int_{B_{p_i}(\delta)} |y - p_i|^{2\alpha_i} \frac{\lambda^{2(1+\alpha)}}{(1 + (\lambda|y - x|)^{2(1+\alpha)})^2} dy,$$

uniformly for $p_i \in \{p_1, \dots, p_m\} \setminus J_\rho$, and for $|x - p_i| \leq \delta^2$. By a change of variables we are left with

$$\int_{B_0(\delta)} \frac{\lambda^{2(1+\alpha)} |y|^{2\alpha_i}}{(1 + (\lambda|y - x|)^{2(1+\alpha)})^2} dy; \quad |x| \leq \delta^2.$$

We next divide the domain into the three sets

$$\begin{aligned}\mathcal{B}_1 &= \{|y| \leq \sqrt{\delta}|x|\}; & \mathcal{B}_2 &= \{\sqrt{\delta}|x| < |y| \leq \delta^{-\frac{1}{2}}|x|\}; \\ \mathcal{B}_3 &= \{\delta^{-\frac{1}{2}}|x| < |y| \leq \delta\}.\end{aligned}$$

In \mathcal{B}_1 we have that $|y - x| = (1 + o_\delta(1))|x|$, which implies

$$\begin{aligned}\int_{\mathcal{B}_1} \frac{\lambda^{2(1+\alpha)}|y|^{2\alpha_i}}{(1 + (\lambda|y - x|)^{2(1+\alpha)})^2} dy &\geq C_\delta^{-1} \frac{\lambda^{2(1+\alpha)}}{(1 + (\lambda|x|)^{2(1+\alpha)})^2} \int_{B_0(\sqrt{\delta}|x|)} |y|^{2\alpha_i} dy \\ &\geq C_\delta^{-1} \frac{\lambda^{2(1+\alpha)}|x|^{2\alpha_i+2}}{(1 + (\lambda|x|)^{2(1+\alpha)})^2}.\end{aligned}$$

In \mathcal{B}_2 $|y|$ is bounded above and below by constants (depending on δ) multiplying $|x|$, and hence with a change of variables we get

$$\begin{aligned}\int_{\mathcal{B}_2} \frac{\lambda^{2(1+\alpha)}|y|^{2\alpha_i}}{(1 + (\lambda|y - x|)^{2(1+\alpha)})^2} dy &\geq C_\delta^{-1} \int_{\lambda(\mathcal{B}_2 - x)} \frac{|x|^{2\alpha_i} \lambda^{2\alpha} dz}{(1 + |z|^{2(1+\alpha)})^2} \\ &\geq C_\delta^{-1} |x|^{2\alpha_i} \lambda^{2\alpha} \int_0^{\frac{\lambda|x|}{C_\delta}} \frac{s ds}{(1 + s^{1+\alpha})^2}.\end{aligned}$$

Integrating we obtain

$$\int_{\mathcal{B}_2} \frac{\lambda^{2(1+\alpha)}|y|^{2\alpha_i}}{(1 + (\lambda|y - x|)^{2(1+\alpha)})^2} dy \geq C_\delta^{-1} |x|^{2\alpha_i} \lambda^{2\alpha} \frac{\lambda^2 |x|^2}{1 + \lambda^2 |x|^2}.$$

Finally in \mathcal{B}_3 we have that $|y - x|^2 = (1 + o_\delta(1))|y|^2$, and therefore from a change of variables we get

$$\begin{aligned}\int_{\mathcal{B}_3} \frac{\lambda^{2(1+\alpha)}|y|^{2\alpha_i}}{(1 + (\lambda|y - x|)^{2(1+\alpha)})^2} dy &\geq C_\delta^{-1} \int_{\mathcal{B}_3} \frac{\lambda^{2(1+\alpha)}|y|^{2\alpha_i} dy}{(1 + (\lambda|y|)^{2(1+\alpha)})^2} \\ &\geq C_\delta^{-1} \lambda^{2(\alpha - \alpha_i)} \int_{C_\delta \lambda |x|}^{\frac{\lambda}{C_\delta}} \frac{s^{2\alpha_i+1} ds}{(1 + s^{2(1+\alpha)})^2}.\end{aligned}$$

Evaluating the last integral we have that

$$\int_{\mathcal{B}_3} \frac{\lambda^{2(1+\alpha)}|y|^{2\alpha_i}}{(1 + (\lambda|y - x|)^{2(1+\alpha)})^2} dy \geq C_\delta^{-1} \lambda^{2(\alpha - \alpha_i)} \frac{1}{1 + (\lambda|x|)^{2+4\alpha-2\alpha_i}}.$$

In conclusion we obtain the estimate

$$\int_{B_{p_i}(\delta)} \frac{|y - p_i|^{2\alpha_i} \lambda^2}{(1 + \lambda^2 |y - x|^2)^2} dy \geq C_\delta^{-1} \left(\frac{\lambda^{2(1+\alpha)} |x|^{2(1+\alpha_i)}}{1 + \lambda^2 |x|^2} + \frac{\lambda^{2(\alpha - \alpha_i)}}{1 + (\lambda|x|)^{2+4\alpha-2\alpha_i}} \right).$$

Using elementary arguments, one can easily check that the last quantity is always greater or equal to $C_\delta^{-1} \lambda^{2\alpha-2\alpha_i}$.

Therefore, adding the three terms in I_ρ we find

$$\begin{aligned}I_\rho(\varphi_{\alpha,\lambda,x}) &\leq 8\pi(1+\alpha)^2 \log \lambda - 2\rho(1+\alpha) \log \lambda - 2\rho(\alpha - \alpha_i) \log \alpha + l.o.t. \\ &= 2 \log \lambda (4\pi(1+\alpha)^2 - (1+\alpha)\rho - \rho(\alpha - \alpha_i)) + l.o.t.\end{aligned}$$

since $\rho > 4\pi(1 + \alpha_i)$ and since $\alpha > \alpha_i$, we get the conclusion. ■

We also have the following result, regarding the concentration properties of the functions $\tilde{h}e^{2\varphi_{\alpha,\lambda,x}}$.

Lemma 5.2. *Let $\varphi_{\alpha,\lambda,x}$ be defined as in Lemma 5.1. Then, for any $x \in \Theta_\rho$,*

$$\frac{\tilde{h}e^{2\varphi_{\alpha,\lambda,x}}}{\int_\Sigma \tilde{h}e^{2\varphi_{\alpha,\lambda,x}}dV_g} \rightarrow \delta_x \quad \text{as } \lambda \rightarrow +\infty.$$

PROOF. Given $\varepsilon > 0$ it is sufficient to show that

$$\frac{\int_{\Sigma \setminus B_x(\varepsilon)} \tilde{h}e^{2\varphi_{\alpha,\lambda,x}}dV_g}{\int_\Sigma \tilde{h}e^{2\varphi_{\alpha,\lambda,x}}dV_g} \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

By the proof of Lemma 5.1 we derived that $\int_\Sigma \tilde{h}e^{2\varphi_{\alpha,\lambda,x}}dV_g \geq C^{-1}\lambda^{2\alpha-2\hat{\alpha}}$, and therefore it is sufficient to check that

$$(30) \quad \frac{\int_{\Sigma \setminus B_x(\varepsilon)} \tilde{h}e^{2\varphi_{\alpha,\lambda,x}}dV_g}{\lambda^{2\alpha-2\hat{\alpha}}} \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

For doing this, we notice that

$$e^{2\varphi_{\alpha,\lambda,x}} \leq C_\varepsilon \lambda^{-2(1+\alpha)} \quad \text{as } \lambda \rightarrow +\infty,$$

and therefore (30) follows immediately. ■

We next define the min-max scheme which is needed to prove Theorem 1.1. We fix $L > 0$ as in Proposition 4.1, and then $\lambda > 0$ so large that $I_\rho(\varphi_{\alpha,\lambda,x}) < -2L$ for $x \in \Theta_\rho$. The latter choice is possible in view of Lemma 5.1.

We then define the set

$$\Lambda_\lambda = \{\varphi_{\alpha,\lambda,x} : x \in \Theta_\rho\}.$$

Next, we consider the topological cone

$$\hat{\Theta}_\rho = (\Theta_\rho \times [0, 1]) / (\Theta_\rho \times \{1\}),$$

where the equivalence relation identifies all the points in $\Theta_\rho \times \{1\}$. Let us introduce next the family of continuous maps

$$\mathcal{H}_{\lambda,\rho} = \left\{ \mathfrak{h} : \hat{\Theta}_\rho \rightarrow H^1(\Sigma) : \mathfrak{h}(x) = \varphi_{\alpha,\lambda,x} \text{ for every } x \in \Theta_\rho \right\},$$

and the number

$$\overline{\mathcal{H}}_{\lambda,\rho} = \inf_{\mathfrak{h} \in \mathcal{H}_\lambda} \sup_{z \in \hat{\Theta}_\rho} I_\rho(\mathfrak{h}(z)).$$

We have then the following result.

Proposition 5.3. *Under the assumptions of Theorem 1.1, if λ is sufficiently large the number $\overline{\mathcal{H}}_{\lambda,\rho}$ is finite. Moreover $\overline{\mathcal{H}}_{\lambda,\rho}$ is a critical value of I_ρ .*

PROOF. If L is as in Proposition 4.1, we show that indeed $\overline{\mathcal{H}}_{\lambda,\rho} > -\frac{3}{2}L$. In fact, suppose by contradiction that there exists a map \mathfrak{h}_0 such that

$$(31) \quad \mathfrak{h}_0 \in \mathcal{H}_{\lambda,\rho} \quad \text{and} \quad \sup_{z \in \hat{\Theta}_\rho} I_\rho(\mathfrak{h}_0(z)) \leq -\frac{3}{2}L.$$

Then Proposition 4.4 applies and gives a continuous map $F_{\lambda,\rho} : \hat{\Theta}_\rho \rightarrow \Theta_\rho$ defined as the composition

$$F_{\lambda,\rho} = \Psi \circ \mathfrak{h}_0.$$

since $\mathfrak{h}_0 \in \mathcal{H}_{\lambda,\rho}$, and hence it coincides with $\varphi_{\alpha,\lambda,\cdot}$ on $\partial\hat{\Theta}_\rho \simeq \Theta_\rho$, by Lemma 5.2 and Remark 3.2 we deduce that

$$(32) \quad F_{\lambda,\rho}|_{\Theta_\rho} \text{ is homotopic to } Id|_{\Theta_\rho}.$$

Here, the homotopy is given by the parameter λ as $\lambda \rightarrow +\infty$.

Let us write as pairs (x, ω) the elements of $\hat{\Theta}_\rho$. If we let ω run between 1 and 0, and we consider the maps $F_{\lambda,\rho}(\cdot, \omega) : \Theta_\rho \rightarrow \Theta_\rho$, we obtain an homotopy between $F_{\lambda,\rho}|_{\Theta_\rho}$ and a constant map. Since by our assumptions Θ_ρ is not contractible, we obtain a contradiction with (32). This proves the first part of the statement.

To check that $\overline{\mathcal{H}}_{\lambda,\rho}$ is a critical level, we use a monotonicity method introduced by Struwe, and which has been used extensively in the study of Liouville type equations. We consider a sequence $\rho_n \rightarrow \rho$ and the corresponding functionals I_{ρ_n} . All the above estimates and results can be worked out for I_{ρ_n} as well with minor changes.

We then define the number $\tilde{\mathcal{H}}_{\lambda,\rho} := \frac{\overline{\mathcal{H}}_{\lambda,\rho}}{\rho}$, which corresponds to the functional $\frac{I_\rho}{\rho}$. It is immediate to see that

$$\rho \mapsto \tilde{\mathcal{H}}_{\lambda,\rho} \quad \text{is monotone ,}$$

and, reasoning as in [27], there exists a subsequence of $(\rho_n)_n$ such that I_{ρ_n} has a solution u_n at level $\overline{\mathcal{H}}_{\lambda,\rho_n}$. Then, applying Theorem 2.5 and passing to a further subsequence, we obtain that u_n converges to a critical point u of I_ρ at level $\overline{\mathcal{H}}_{\lambda,\rho}$. \blacksquare

Remark 5.4. (a) In [19] the Leray-Schauder degree of (4) is being computed, using refined blow-up analysis and Lyapunov-Schmidt reductions, in the spirit of [17], [18]. Taking into account the result in [45], we speculate that there should be a relation between the degree and the Euler characteristic of the set $\Sigma \setminus J_\rho$. Anyway, the min-max methods might give existence (or even multiplicity, see [25], [26]) results even when the total degree of the equation vanishes. On the other hand, we have to avoid some critical values of ρ which are instead treatable via blow-up analysis.

(b) Further results are discussed in [5], where the case of arbitrary positive α 's is discussed for surfaces with positive genus. In this case one can exploit the non simply connectedness of the surfaces and avoid to use of the improved inequality we derived here. However the latter should be necessary to treat the case with singularities of different signs (or to characterize the homology of low sublevels of I_ρ , as in [47]). The case of the sphere for larger values of ρ will be treated in a forthcoming paper, combining our approach with some techniques in [29] and a topological argument. In [8] and [14] the case of negative α 's is studied: in this situation one can combine Troyanov's result with Chen-Li's inequality.

REFERENCES

- [1] Aubin T., *Meilleures constantes dans le theoreme d'inclusion de Sobolev et un theoreme de Fredholm non lineaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. 32 (1979), 148-174.
- [2] Bartolucci D., *A sup+cinf inequality for Liouville-type equations with singular potentials*, preprint, 2007.
- [3] Bartolucci D., *A sup+cinf inequality for the equation $-\Delta u = \frac{V}{|x|^{2\alpha}} e^u$* , preprint, 2007.

- [4] Bartolucci D., Chen C.C., Lin C.S., Tarantello G., *Profile of blow-up solutions to mean field equations with singular data*, Comm. Part. Diff. Eq. 29 (2004), 1241-1265.
- [5] Bartolucci D., De Marchis F., Malchiodi A., *Supercritical conformal metrics on surfaces with conical singularities*, preprint, 2010.
- [6] Bartolucci D., Lin C.S., *Uniqueness results for mean field equations with singular data*, Comm. Part. Diff. Eq. 34 (2009), no. 7-9, 676-702.
- [7] Bartolucci D., Lin C.S., Tarantello G., *Uniqueness and symmetry results for solutions of a mean field equation on S^2 via a new bubbling phenomenon*, preprint, 2010.
- [8] Bartolucci D., Montefusco E., *Blow-up analysis, existence and qualitative properties of solutions of the two-dimensional Emden-Fowler equation with singular potential*, Math. Meth. Appl. Sci. 30 (2007), 2309-2327.
- [9] Bartolucci D., Tarantello G., *The Liouville equation with singular data: a concentration compactness principle via a local representation formula*, J. Diff. Eq. 185 (2002) 161-180.
- [10] Bartolucci D., Tarantello G., *Liouville type equations with singular data and their application to periodic multivortices for the electroweak theory*, Comm. Math. Phys. 229 (2002) 3-47.
- [11] Brezis H., Li Y.Y., Shafrir I., *A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities*, J. Funct. Anal. 115 (1993), 344-358.
- [12] Brezis H., Merle F., *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions* Commun. Partial Differ. Equations 16-8/9 (1991), 1223-1253.
- [13] Cabré X., Lucia M., Sanchon M., *A mean field equation on a torus: one-dimensional symmetry of solutions*, Comm. Part. Diff. Eq. 30, no. 7-9 (2005), 1315-1330.
- [14] Carlotto A., to appear.
- [15] Chang S.Y.A., Yang, P. C., *Prescribing Gaussian curvature on S^2* , Acta Math. 159, no. 3-4 (1987), 215-259.
- [16] Chanillo S., Kiessling M., *Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry*, Comm. Math. Phys. 160 (1994), 217-238.
- [17] Chen C.C., Lin C.S., *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. 55-6 (2002), 728-771.
- [18] Chen C.C., Lin C.S., *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. 56-12 (2003), 1667-1727.
- [19] Chen C.C., Lin C.S., *A degree counting formulas for singular Liouville-type equation and its application to multi vortices in electroweak theory*, in preparation.
- [20] Chen C.C., Lin C.S., Wang G., *Concentration phenomena of two-vortex solutions in a Chern-Simons model*, Ann. Scuola Norm. Sup. Pisa, Vol. III, issue 2 (2004), 367-397.
- [21] Chen W., Li C., *Prescribing Gaussian curvatures on surfaces with conical singularities*, J. Geom. Anal. 1-4, (1991), 359-372.
- [22] Chen W., Li C., *Qualitative properties of solutions of some nonlinear elliptic equations in R^2* , Duke Math. J. 71-2 (1993), 427-439.
- [23] Chou K.S., Wan T.Y.H., *Asymptotic radial symmetry for solutions of $\Delta u + \exp u = 0$ in a punctured disc*, Pacific J. of Math. 163 (1994), 269-276.
- [24] Del Pino M., Esposito P., Musso M., *Two-dimensional Euler flows with concentrated vorticities*, preprint, 2008.
- [25] De Marchis F., *Multiplicity result for a scalar field equation on compact surfaces*, Comm. Part. Diff. Eq. 33, no. 10-12 (2008), 2208-2224.
- [26] De Marchis F., *Generic multiplicity for a scalar field equation on compact surfaces*, preprint.
- [27] Ding W., Jost J., Li J., Wang G., *Existence results for mean field equations*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 16-5 (1999), 653-666.
- [28] Djadli Z., *Existence result for the mean field problem on Riemann surfaces of all genres*, Comm. Contemp. Math., 10, no. 2 (2008), 205-220.
- [29] Djadli Z., Malchiodi A., *Existence of conformal metrics with constant Q -curvature*, Ann. of Math., 168 (2008), no. 3, 813-858.
- [30] Dolbeault J., Esteban M.J., Tarantello G., *The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008), no. 2, 313-341.
- [31] Druet O., Robert F., *Bubbling phenomena for fourth-order four-dimensional pdes with exponential growth*, Proc. Amer. Math. Soc. 134 (2006), no. 3, 897-908.
- [32] Dunne G., *Self-dual Chern-Simons Theories*, Lecture Notes in Physics (1995).

- [33] Esposito, P., *Blowup solutions for a Liouville equation with singular data*, SIAM J. Math. Anal. 36 (2005), no. 4, 1310-1345.
- [34] Graham C.R., Jenne R., Mason L.J., Sparling G., *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc. 46-3 (1992) 557-565.
- [35] Hong J., Kim Y., Pac P.Y., *Multivortex Solutions of the Abelian Chern-Simons Theory*, Phys. Rev. Lett. 64 (1990), 2230-2233.
- [36] Jackiw R., Weinberg E.J., *Selfdual Chern-Simons vortices*, Phys. Rev. Lett. 64 (1990), 2234-2237.
- [37] Lai C.H. (ed.), *Selected Papers on Gauge Theory of Weak and Electromagnetic Interactions*, World Scientific, Singapore, 1981.
- [38] Li Y.Y., *Harnack type inequality: The method of moving planes*, Commun. Math. Phys. 200-2, (1999), 421-444.
- [39] Li Y.Y., Shafrir I., *Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two*, Indiana Univ. Math. J. 43-4 (1994), 1255-1270.
- [40] Lin C.S., Lucia M., *Uniqueness of solutions for a mean field equation on torus*, J. Diff. Eq. 229 (2006), no. 1, 172-185.
- [41] Lin C.S., Lucia M., *One-dimensional symmetry of periodic minimizers for a mean field equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6 (2007), no. 2, 269-290.
- [42] Lin C.S., Wang C.L., *Elliptic functions, Green functions and the mean field equations on tori*, Ann of Math., to appear.
- [43] Lucia M., *A deformation lemma with an application to a mean field equation*, Topol. Methods Nonlinear Anal. 30 (2007), no. 1, 113-138.
- [44] Lucia M., Zhang L., *A priori estimates and uniqueness for some mean field equations*, J. Diff. Eq. 217 (2005), no. 1, 154-178.
- [45] Malchiodi A., *Compactness of solutions to some geometric fourth-order equations*, J. Reine Angew. Math., 594 (2006), 137-174.
- [46] Malchiodi A., *Topological methods for an elliptic equations with exponential nonlinearities*, Discrete Contin. Dyn. Syst. 21 (2008), no. 1, 277-294.
- [47] Malchiodi A., *Morse theory and a scalar field equation on compact surfaces*, Adv. Diff. Eq. 13 (2008), no. 11-12, 1109-1129.
- [48] Moser J., *A sharp form of an inequality by N.Trudinger*, Indiana Univ. Math. J. 20 (1971), 1077-1091.
- [49] Moser J., *On a nonlinear problem in differential geometry*, Dynamical Systems (M. Peixoto ed.), Academic Press, New York, 1973, 273-280.
- [50] Nagasaki K., Suzuki T., *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities*, Asymptotic Anal. 3 (1990), no. 2, 173-188.
- [51] Prajapat J., Tarantello G., *On a class of elliptic problems in R^2 : Symmetry and Uniqueness results*, Proc. Roy. Soc. Edinburgh 131A (2001), 967-985.
- [52] Struwe M., *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math. 160 -1/2(1988), 19-64.
- [53] Struwe M., Tarantello G., *On multivortex solutions in Chern-Simons gauge theory*, Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8)-1, (1998), 109-121.
- [54] Tarantello G., *A quantization property for blow up solutions of singular Liouville-type equations*, J. Func. Anal. 219 (2005), 368-399.
- [55] Tarantello G., *An Harnack inequality for Liouville-type equations with singular sources*, Indiana Univ. Math J. 54 n.2 (2005), 599-615.
- [56] Tarantello G., *Self-Dual Gauge Field Vortices: An Analytical Approach*, PNLDE 72, Birkhäuser Boston, Inc., Boston, MA, 2007.
- [57] Bartolucci D., Lin C.S., Tarantello G., *Analytical, geometrical and topological aspects of a class of mean field equations on surfaces*, Discrete and Continuous Dynamical systems, vol 28, n.3 (2010).
- [58] Troyanov M., *Prescribing Curvature on Compact Surfaces with Conical Singularities*, Trans. A.M.S., Vol. 324, No. 2. (1991), 793-821.
- [59] Yang Y., *Solitons in Field Theory and Nonlinear Analysis*, Springer Monographs in Mathematics, Springer, New York, 2001.
- [60] Zhang L., *Asymptotic behavior of blowup solutions for elliptic equations with exponential nonlinearity and singular data*, Comm. Contemp. Math., 11, No. 3 (2009) 395-411.

SISSA, VIA BONOMEA 265, 34136 TRIESTE (ITALY) AND DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,
UNIVERSITY OF GRANADA, 18071 GRANADA (SPAIN).

E-mail address: `malchiod@sissa.it`, `daruiz@ugr.es`